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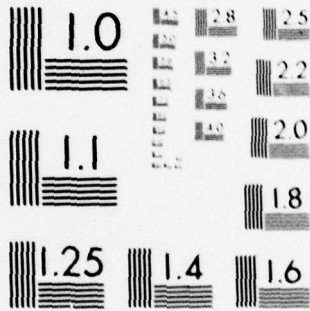ESL-TM-817   AFOSR-TR-78-1031   NL

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A through understanding of analog-to-digital coding is a first step toward a new theory of digital control. As opposed to present theories which assume continuous control levels, a new theory of direct finite-level digital control may lead to more efficient implementations and better performance. Unfortunately, the mathematics of conversion of coding continuous waveforms into finite-level waveforms (acceptable to computers) is not well understood.

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Ultimately we want to understand the interaction of continuous and digital feedback, since new insights may be gained in the design of digital controllers.

But before systematic design procedures can be found, we must throughly understand the interaction of digital machines with continuous plants through the coding and decoding processes.

As a preliminary mathematical model we can assume each element on the feedback loop can be represented as a functional mapping from the set of all possible input waveforms to the set of all output waveforms.

For a meaningful theory the class of input and output waveforms and the kind of functionals between them must be restricted to reflect the properties of physical elements themselves.

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(10) Steven N. Jones

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1. Motivation

A thorough understanding of analog-to-digital coding is a first step toward a new theory of digital control [0]. As opposed to present theories which assume continuous control levels, a new theory of direct finite-level digital control may lead to more efficient implementations and better performance. Unfortunately, the mathematics of conversion of coding continuous waveforms into finite-level waveforms (acceptable to computers) is not well understood.

Consider a typical feedback situation illustrated in Figure 1, where a continuous plant is controlled by a digital computer. Because the continuous waveform from the plant must be converted to digital, a coder is necessary, and a decoder for digital-to-analogy conversion.

Ultimately we want to understand the interaction of continuous and digital feedback, since new insights may be gained in the design of digital controllers. For example, it has been shown [1] that a first-order linear system can be stabilized by a digital system as simple as a flip-flop. Obviously this implementation is much less costly than a sampled-data system with a digital multiplier, which would be assumed in present theories of feedback control.

But before systematic design procedures can be found, we must throughly understand the interaction of digital machines with continuous plants through the coding and decoding processes.

As a preliminary mathematical model we can assume each element on the feedback loop can be represented as a functional mapping from the set of all possible input waveforms to the set of all output waveforms. In particular we are, therefore, studying a very general coder, one that might have memory and whose output might reflect very complex properties of the continuous input waveform.

For a meaningful theory the class of input and output waveforms and the kind of functionals between them must be restricted to reflect the properties of physical elements themselves. When such restrictions, or properties are given to the systems and spaces being studied, the complex systems can often be decomposed into simpler units and this new understanding results in simplified design procedures and stronger statements about the limitations of such systems. In the cases of continuous plants and computer controllers, these simplifications have been made and are very useful (Continuous system, [2] and [3]; digital [4]). Less is known about asynchronous computer controllers [5]. But there is little known about general coding functionals.

2. <u>Problem Statement</u>

Let $\mathcal{D} \subset PC(R; R^n)$ (that is, $\mathcal{D}$ is subset of all piecewise continuous functions from R to R^n), and let $\mathcal{R} \subset PC(R; X)$ where x is a finite set. The functional

S: $\mathcal{D} \rightarrow \mathcal{R}$

is a <u>coder</u>. S is causal if

$$P_T f_1 = P_T f_2 \text{ implies } Sf_1 = Sf_2$$

where $f_1, f_2 \in \mathcal{D}$ and $P_T f = \begin{cases} f & t \le T \\ 0 & 0 > T \end{cases}$

Of course many of the coding functionals which meet the above requirements do not correspond to the physical behavior of any real device. This suggests two problems: first, what property can be formulated which captures the implementable of a coders; second, can this property (perhaps strengthened) be shown to imply a canonical structure of such coding devices. Such a decomposition might involve a continuous linear part, a non-linear memoryless threshold, and a nonlinear digital part.

3. <u>Summary of Present Work</u>

We begin with an example to illustrate the difficulties in formulating a plausible realizability criterion.

3.1 <u>Defn.</u>   Let $\theta_0: R \to \{0,1\}$ be defined by

$$\theta_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 1 \end{cases}$$

Let $S_{\theta_0}: PC(R,R) \to RC(R^+,\{0,1\})$ be the elementary <u>0-threshold coder</u> defined by

$$S_{\theta_0}(f) = \theta_0 \cdot f$$

S_{θ_0} is a memoryless coder and very simple to implement, so any reasonable class of mathematically-defined   coders must include it. Obviously linearity cannot be assumed; also, the 0-threshold coder is <u>discontinuous</u> in the usual sense. This is illustrated in the following example; a few preliminary definitions will be given for completeness.

3.2 <u>Defn.</u>   Let Y be a set and let d be a metric on Y^R. We write

$$\lim_d f_i = f$$

for a sequence $f_i \in Y^R$ and $f \in Y^R$ iff for any $T > 0$ and any $\epsilon > 0$ there is an $N \in Z^+$ such that

$$d(P_T f_i, P_T f) < \epsilon \qquad \text{when } i > N.$$

3.3 <u>Defn.</u>   A coder $S: PC(R^+,R) \to PC(R,X)$ is $\underline{d_1 d_2 \text{ continuous}}$ iff

$$\lim_{d_1} f_i = f \qquad (f_i \in PC(R^+,R))$$

implies $\lim_{d_2} Sf_i = Sf$

3.4 Example. Let

$$f_i(t) = \frac{-1}{i} \qquad t \in R^+, \quad i \geq 1$$

Then under most any metric d_1 (for example, Lesbegue integration)

$$\lim_{d_1} f_i = 0$$

which implies S_{θ_0} is discontinuous since for any metric d_2

$$d_2(0,1) \neq 0$$

where

$$0 = \lim_{d_2} S_{\theta_0} f_i$$

$$1 = S_{\theta_0} \lim_{d_1} f_i \qquad \square$$

Even though S_{θ_0} is discontinuous in the usual sense, it is still reasonable to look for some kind of continuity. The idea we have pursued is continuity in _transition times_. This is best illustrated by an example.

3.5 Example. Let $f_i = \sin(i + \frac{1}{i})t$ $i \geq 1$, $t \geq 0$. The result of passing these waveforms through S_{θ_0} can be determined from the zero-crossing times of the input. For example,

$$f_2^+(0) = (0, \frac{3\pi}{2}, \frac{6\pi}{2}, \frac{9\pi}{2}, \ldots)$$

where $f_2^+(0)$ means the series of zero-crossing times made by f_2 when $t \geq 0$.

A similar series could be defined on the binary output as the series of transition times, either 1 to 0 or 0 to 1; obviously this series would be the same as $f_2^+(0)$ and we could write

$$(S_{\theta_0} f_2)^+ = f_2^+(0) = (0, \frac{3\pi}{2}, \frac{6\pi}{2}, \ldots)$$

Letting $f_2^+(0)_k$ be the kth element in the series, define

$$\rho_1(f,g) = \sum_{k=1}^{\infty} \left| f^+(0)_k - g^+(0)_k \right|$$

Looking at the transition times

$$f_i^+(0) = (0, \frac{\pi i}{1+i}, \frac{2\pi i}{1+i}, \ldots)$$

we see they will converge under ρ to $\sin^+(0) = (0, \pi, 2\pi, \ldots)$:

$$\lim_\rho f_i = \sin t$$

and since $(S_{\theta_0} f_i)^+ = f_i^+(0)$,

$$\lim_\rho S_{\theta_0} f_i = S_{\theta_0} \lim f_i$$

which shows a limited kind of continuity of transition times in this particular case.

There are two problems in making this idea rigorous. First, we must be sure that the functions under question have well defined transition sequences, not infinitely many transitions in a short period. Second,

we must define a more general metric ρ so that a reasonable class of thresholds are continuous. Sequences like the one illustrated in Example 3.4 should be divergent.

3.6 Example

Let $f(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{x}} \sin \frac{1}{x} & t > 0 \end{cases}$

It can be shown that f is C^∞. Yet there is no first zero-crossing for $t > 0$.

Thus C^∞ is still too large a class of functions to meet our requirements. We now introduce a class $F_{R^+,Y}$ of <u>normal functions</u> from R^+ to a set Y, which have well defined transition times, closure under composition, and lead naturally to a notion of continuity which includes all thresholds.

Normality Spaces

We will first define a class of <u>normal sets</u> on a space X in analogy to a class of open sets in a topology. The idea is to then let f: $x \to Y$ be a normal set if the inverse image of a normal set is normal, again in analogy to continuity. Given a few additional properties, and by choosing our class of normal sets properly, we will show a class of normal functions are closed under composition and have well-defined transition times.

3.7 <u>Defn</u>. A triple (X, ρ_x, N_x) is a <u>normality space</u> if

i) ρ_x is a metric on X

ii) N_x is a subset of the Borel sets generated by ρ_x, called the set of normal sets

iii) If $A, B \in N_x$ then

    a) A is bounded

    b) $A \cup B, A - B \in N_x$

iv) Every bounded set in X is contained in a normal set

Note that $A \cap B$ is also normal.

3.8 <u>Defn.</u>   An interval in R is a subset (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ where $-\infty < a \leq b < \infty$.

3.9 <u>Defn.</u> $N_R^0 = \{S \mid S$ is a <u>finite</u> union of disjoint intervals in R$\}$.

3.10 <u>Prop.</u> (R, d, N_R^0) is a normality space if $d(a,b) = |a-b|$.

3.11 <u>Defn.</u>   Let $f: R \to Y$ where Y is a normality space. f is <u>normal</u> iff

    a) $f(S)$ is bounded if S is bounded for any $S \subseteq R$

    b) For any normal $T \in N_Y$, $A \in N_R^0$, $A \cap f^{-1}(T)$ is normal.

3.12 <u>Prop.</u>   If $R \xrightarrow{f} R \xrightarrow{q} X$ are normal functions on normality spaces R, X, then $q f$ is normal.

<u>Proof:</u> From the above definition part a) is obviously satisfied.

Let A, T be as in part b). Since A is normal, A is bounded, hence f(A) is bounded, and there is some $S \in N_Y$ s.t. f(A)   S. Then

$$A \cap f^{-1}(S \cap g^{-1}(T)) = A \cap f^{-1} \quad g^{-1}(T) \text{ and } A \cap f^{-1} g^{-1}(T) \text{ must}$$

be normal as required. □

We can use Prop. 3.12 to define a class of normal thresholds which are guaranteed to take normal input functions to normal output functions. Once this feature is guaranteed, it is possible to define continuity over the set of normal functions, which will be done in the next section.

First, suppose X is a finite set of real numbers. Let X also be a normality space by taking every subset open and normal; let θ: R → X be normal. Then the normal θ threshold is the functional

$$S_\theta : PC(R^+, R) \to PC(R^+, X)$$

where $S_\theta(f) = \theta \cdot f$

By Proposition 3.12 $S_\theta(f)$ will be normal if f is: thus any normal θ threshold will have the desired property.

Proof: Immediate from Proposition 3.12.

The real power of our definitions now emerges as we show that all normal functions can be represented as in example 3.5.

3.14 Defn. Let $F_{R^+,Y}$ be the set of all normal functions from R^+ to normality space Y which are right-continuous.

3.15 <u>Proposition</u>   Let $f \in F_{R^+, Y}$ and $y \in Y$. Then $P_T f$ $(T > 0)$ takes on the value y for a finite number of invervals.

<u>Proof:</u> Since $\{y\}$ is normal, so is $[0,T] \cap f^{-1}(y)$.

We can now proceed to define the <u>normal inverse</u> of a function on a rigorous basis.

3.16 Defn.   Let $f \in F_{R^+, Y}$. Let $T \geq 0$ be arbitrary.

1. Define for fixed T, $f_i^T : Y \quad R^+$, $i \geq 1$, by

$$f_i^T(y) \quad = \quad \begin{array}{l} \text{the left endpoint of the ith interval in} \\ f^{-1}(y) \cap [0,T] \text{ if there is an ith interval} \\ \infty \text{ otherwise} \end{array}$$

2. Define $f_i^+(y) \quad = \quad \inf_T \{f_i^T(y)\}$

3. The <u>normal inverse</u> of f is the series of functions $\{f_i^+\}$ $i = 1, 2, \ldots$.

3.17 <u>Example 1.</u>   Calculate $\{\sin_i^+(t)\}$. It can be checked that $\sin(\cdot)$ is normal and $\sin \in F_{R^+, R}$. Figure (2) shows graphically the normal inverse. It can be seen that

$$\sin_i^+(t) \quad = \quad \infty \quad \text{if } |t| > 1$$

and, for example,

$$\sin_i^+(t) \quad = \quad 2\pi \cdot [\frac{i}{2}] + (-1)^{(i+1)} \sin^{-1}(t) \text{ when } 0 \leq t < 1$$

<u>Example 2.</u> Let sq: $R^+ \to \{0,1\}$ be a square wave of period 1 (Figure 3). Then sq $\in F_{R^+, \{0,1\}}$ and

$$sq_i^+(1) = i$$

$$sq_i^+(0) = i + \frac{1}{2}$$
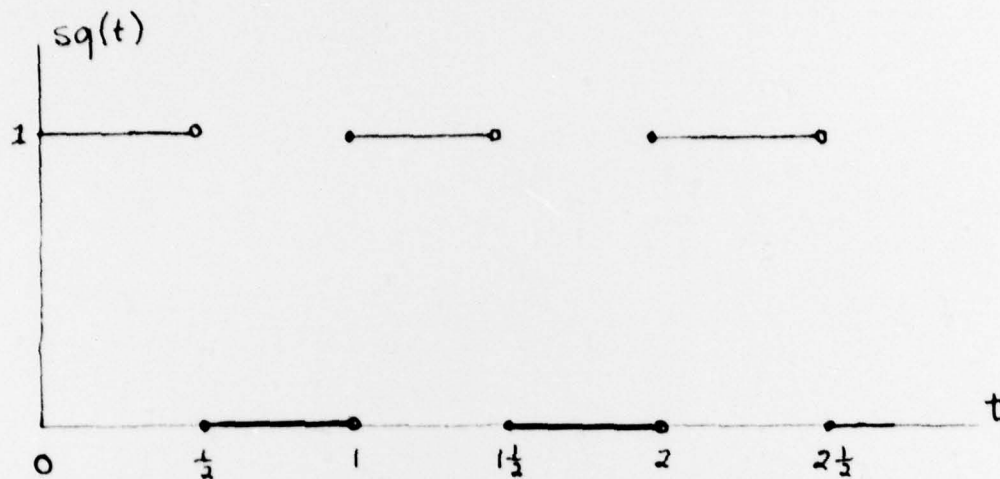
ρ-Continuity

Armed with the normal inverse, we can now completely formalize the idea of Example 3.5.

__3.18 Defn.__   Let $f, g \in F_{R,Y}$.

$\rho(f,g) = \sup |f_i^+ - g_i^+|$ where $\infty - \infty$ is taken to be zero.

__3.19 Proposition:__ ρ is a metric.

__3.20 Proposition:__ Let S_θ be a threshold as in Corollary 3.13.   Then

S_θ is ρ-continuous.

So we have achieved our goal of defining a notion of continuity which includes thresholds. But is it really a useful notion? To get a feel for ρ-continuity we discuss briefly some of its properties.

3.21 Proposition.

1. Let $d(f,g) = \int |f - g|$. Then $\lim_d f_i = f$ if

   $\lim_\rho f_i = f$.

2. If S_1, S_2 are ρ-continuous, so is $S_1 \cdot S_2$.

ρ-continuity also seems to be connected to circuit realizability, as we originally hoped. The following examples illustrate some ρ-continuous coders that are realizable with the configuration on Figure (4): a linear, continuous filter, thresholds, and a digital computer which feeds back to the linear system.

3.22 Example ρ-continuous coders

1. <u>Zero crossing detection</u>. Let D $F_{R^+, \{0,1\}}$ be the set of all normal continuous functions. The zero-crossing detector flips output whenever the input crosses through zero. Figure 5 that a simple threshold will serve as realization.

2. <u>Frequency coder</u>. Note that every $F_{R^+, R}$ is Lesbegue integrable, since the inverse of any half-open interval is a finite union of intervals,

which is measurable. So we can define a <u>frequency coder</u> as

$$S\ f(t)\ =\ \theta_0[\sin \int_0^t (f_0 + f(t))dt]$$

which is ρ-continuous. As an idealized element with which this coder
could be realized, we define the <u>resettable integrater</u> (Figure 6) to be
a two input system which integrates the first input and resets to zero-
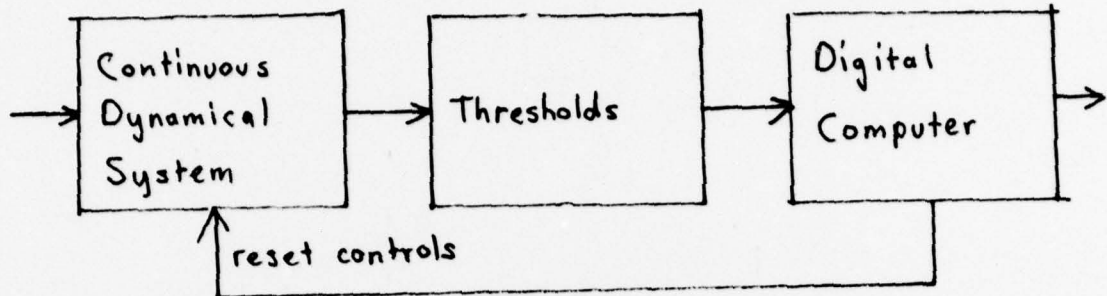state when the second input makes a transition.
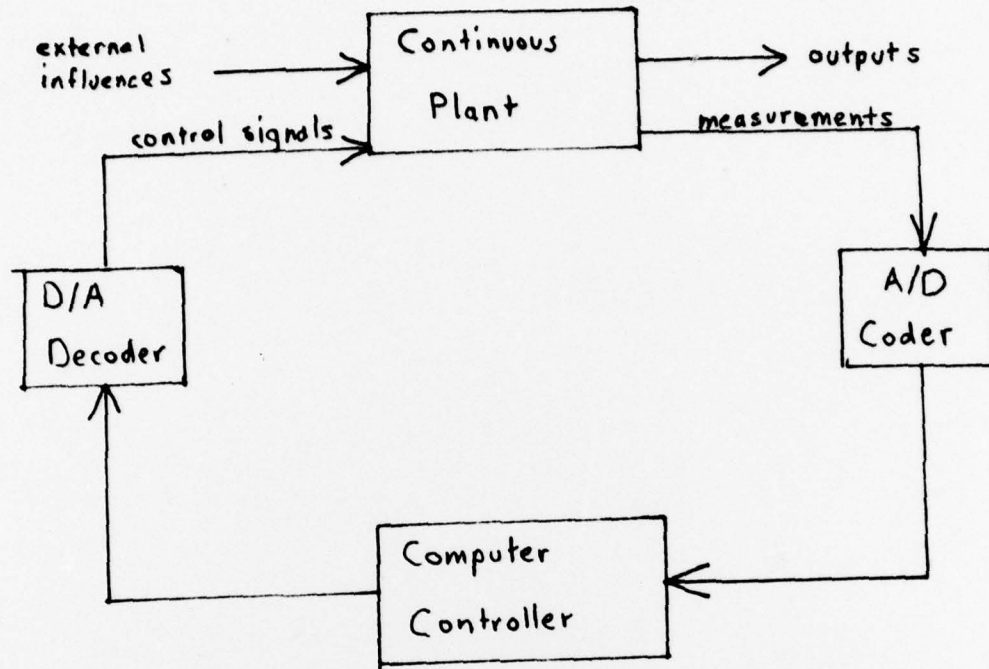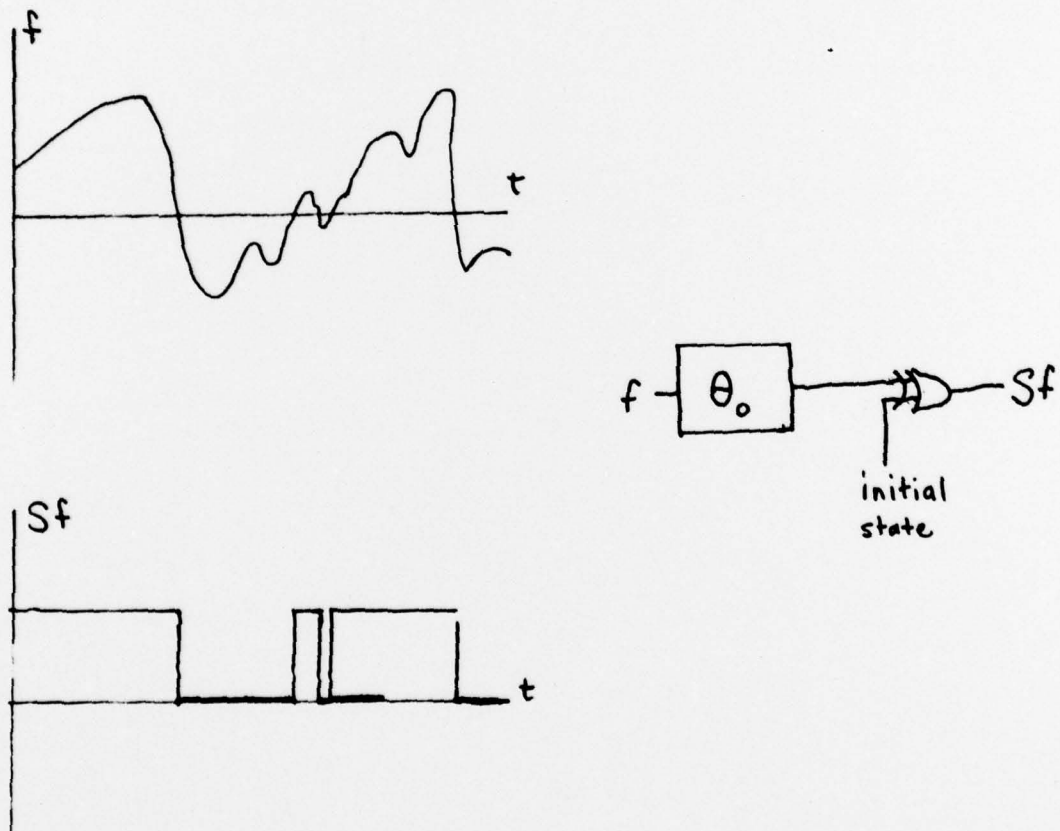
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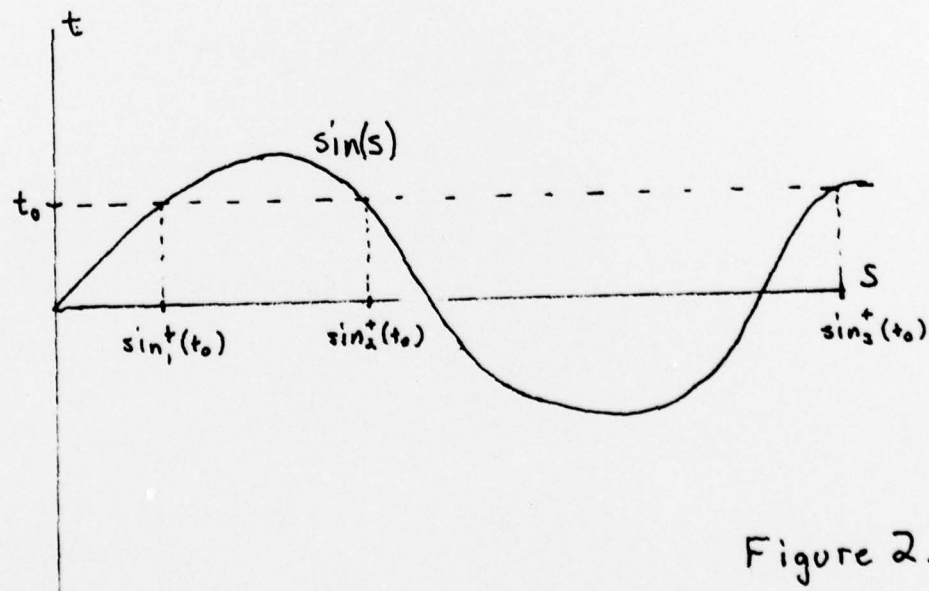
Figure 2.

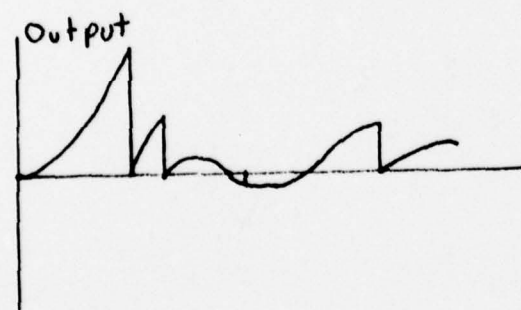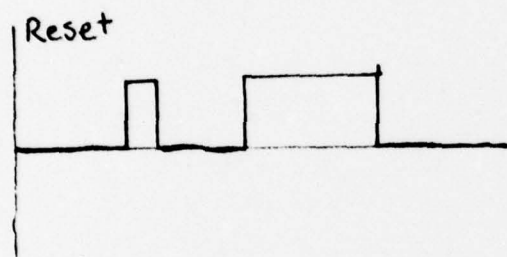Figure 4. Coder Realization
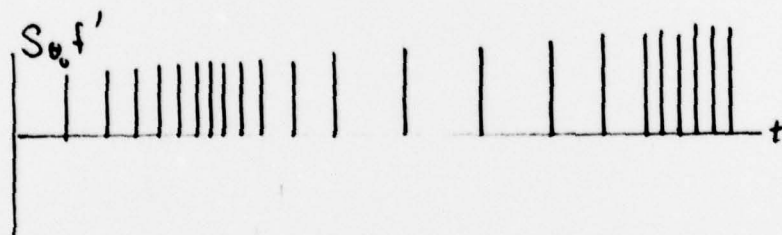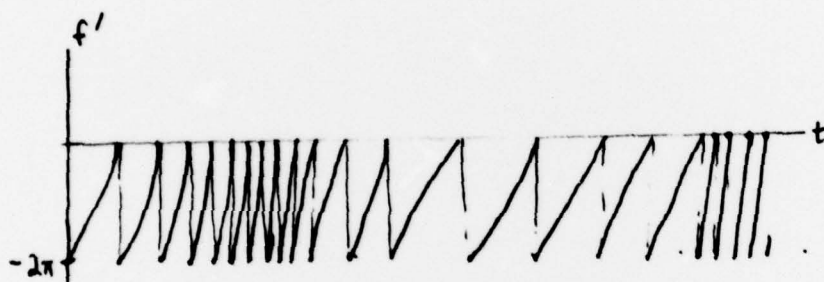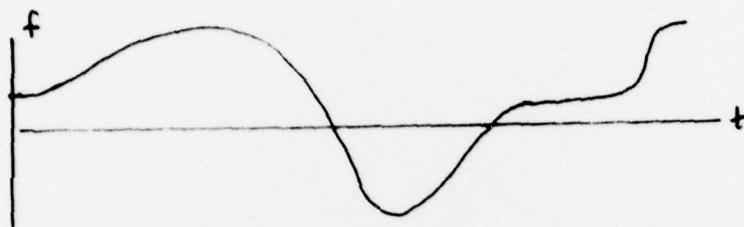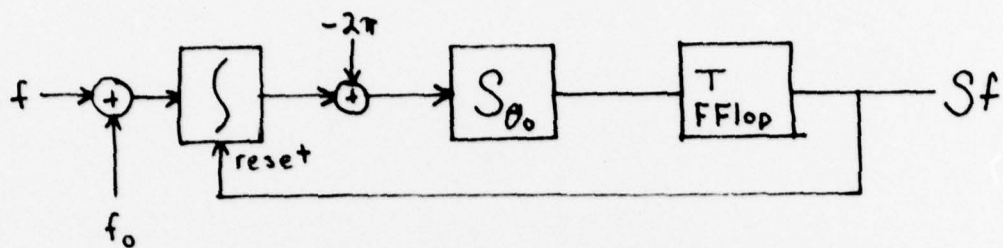

Figure 5. Zero-crossing Detector
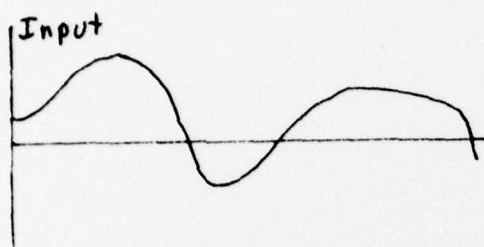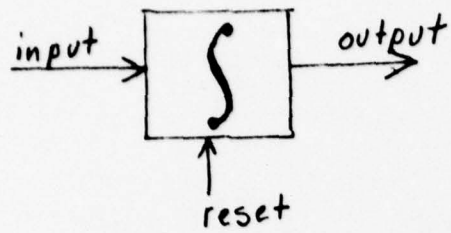
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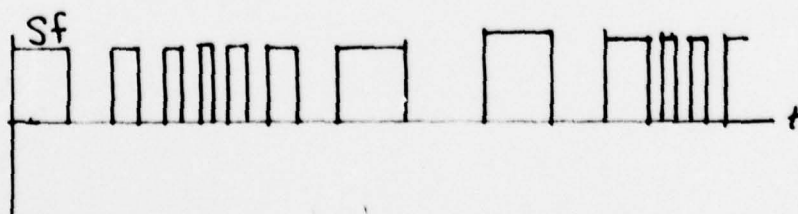
Figure 7.

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